Comment on "Renormalization-group theory for the phase-field crystal equation"

Y. Shiwa
Statistical Mechanics Laboratory, Kyoto Institute of Technology, Matsugasaki, Sakyo-ku, Kyoto 606-8585, Japan
(Received 1 July 2008; published 6 January 2009)

Athreya, Goldenfeld, and Dantzig [Phys. Rev. E **74**, 011601 (2006)] claim that the current implementation of the renormalization-group method neglects the proper ordering of renormalization and differentiation. Their analysis is, however, based on the wrong multiple-scales method results.

DOI: 10.1103/PhysRevE.79.013601 PACS number(s): 81.15.Aa, 81.16.Rf, 05.10.Cc, 61.72.Cc

The phase-field crystal (PFC) model [1] is the continuum approach that is promising in successfully treating many nonequilibrium dynamics arising during materials processing. On the other hand, the amplitude equation approach that describes the slowly varying amplitudes of the order parameter field always plays an important role in theories of pattern formation in general outside of equilibrium. Therefore it is quite natural that the amplitude equation (AEq) has been proposed [2] as a coarse-grained version of the PFC model. We shall call the proposed equation the heuristic AEq since the method to derive it is phenomenological and rather heuristic.

In a recent paper [3], Athreya, Goldenfeld, and Dantzig (AGD) employed various singular perturbation methods to see if the heuristic AEq could be derived from systematically coarse graining the PFC equation. More specifically they used the multiple-scales (MSs) method and variants of the renormalization group (RG). They assume that the criterion upon which to test the accuracy of a theory is the method of MSs. Thus if one method yields the result which is closer to the MSs result than the other, then that method is deemed more correct. The net outcome of their calculations is that none of the current RG methods agrees with the MSs solution.

Confronted with this difficulty and in order to get the RG calculation to agree with the MSs calculation, AGD claimed that one has to depart from the conventional RG procedure originally developed by Chen *et al.* in [4]. Namely, operator ordering in the RG procedure is to be treated properly, and that the amplitude must be renormalized before the differential operation is performed upon the amplitude in exactly the opposite order of operations which is employed in the current RG methods [4–6]. They then showed that, with this remedy implemented, the (proto-)RG method produces the same answer as the method of MSs; to this finding we will return below. To justify the generality of the proposed RG prescription, they then used the Van der Pol oscillator as an example. The explicit results of the proto-RG which is modified as prescribed as above and the MSs calculations were given, and showed that they agreed with each other.

However, the MSs solution given by AGD for the Van der Pol problem omitted a needed extra time scale. In fact, the three-time expansion is necessary to kill the secular terms that appear in their calculations [7], while they erroneously used the two-timing MSs expansion by starting their calculations with the statement: "It is known that the scaling $\tau = \epsilon t$ works for this problem [30]" ([30] being the reference cited in [3]). Redone with the correct number of time scales, the amplitude equation obtained from the MSs method becomes

$$\frac{dA}{dt} = \epsilon\frac{A}{2}(1-|A|^2) - \epsilon^2\frac{i}{8}A\left(1-4|A|^2+\frac{7}{2}|A|^4\right) + O(\epsilon^3), \tag{1}$$

which agrees with the unmodified proto-RG result [5]. Thus the very example that AGD have chosen for their calculations shows that the conventional RG methods should not be modified as advocated by them, for ordinary differential equations, to say the least. Moreover, it is rather an easy exercise to prove that all the conventional RG methods give the identical result, which agrees exactly with the MSs solution (1).

The MSs result for the PFC equation due to AGD has overlooked an important physics. We wish to point out here that the conservation law inherent in the PFC equation implies the existence of neutral modes at zero wave number. The important point to remember is that in the vicinity of the instability to a cellular structure of finite wave number (k_c), the amplitude of slow neutral modes couples to that of the critical modes at k_c and modifies the dynamics significantly. The well-known example is the case of vertical vorticity modes in Rayleigh-Bénard convection with stress-free boundary conditions [8]. Unfortunately, the MSs solution by AGD fails to take these slow modes into account and hence cannot be right. Indeed, the standard MSs analysis shows that the neutral mode generates, e.g., an additional term proportional to

$$(1-\mathcal{L}_{1D})AB \tag{2}$$

to the MSs solution of AGD [i.e., Eq. (46) in [3]; we have used the same notations as therein], where B is the amplitude of the neutral mode. See also Ref. [23] cited by AGD [3]. As already described above, AGD employed the method of MSs as a benchmark for accuracy of theory. Since the MSs solution that AGD employed is thus incorrect, we are inevitably led to the conclusion that the proposed amendment to the current RG methods is unfounded. We remark in closing that although the method of MSs thus could not justify the heuristic AEq, to reproduce the correct MSs solution alluded to above by the now standard (or revised) RG method is another story from the context of the present discussion. Certainly, the RG calculation of AGD itself needs to be improved to take the zero mode into account before any conclusion can be reached.

[1] K. R. Elder and M. Grant, Phys. Rev. E **70**, 051605 (2004).
[2] N. Goldenfeld, B. P. Athreya, and J. A. Dantzig, Phys. Rev. E **72**, 020601(R) (2005).
[3] B. P. Athreya, N. Goldenfeld, and J. A. Dantzig, Phys. Rev. E **74**, 011601 (2006).
[4] L.-Y. Chen, N. Goldenfeld, and Y. Oono, Phys. Rev. E **54**, 376 (1996).
[5] K. Nozaki and Y. Oono, Phys. Rev. E **63**, 046101 (2001); for corrected treatment of the Swift-Hohenberg equation, see K. Nozaki, Y. Oono, and Y. Shiwa, Phys. Rev. E **62**, R4501 (2000). Note also that in the former reference Eq. (3.42) contains a minor calculational error and should read $\frac{dA_R}{d\tau} = \epsilon \frac{A_R}{2}(1-|A_R|^2) - \epsilon^2 \frac{i}{8} A_R (1-4|A_R|^2 + \frac{7}{2}|A_R|^4) + O(\epsilon^3)$.
[6] B. Mudavanhu and R. E. O'Malley, Stud. Appl. Math. **107**, 63 (2001).
[7] A. Nayfeh, *Perturbation Methods* (Wiley, New York, 1973), Sec. 6.2.2.
[8] A. Zippelius and E. D. Siggia, Phys. Fluids **26**, 2905 (1983).